Submersion of a Markov Process

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Abstract

This note describes the transformation of a particular Gaussian process in $\mathbb{R}^3$ by a submersion to $\mathbb{R}^2$. The resulting process (x, y) is a time-homogeneous Markovian martingale for which (x_t, y_t) is Gaussian for every $t > 0$ and the covariance matrix of (x_t, y_t) is exponentially increasing in t.

Keywords: Markov process, martingale, submersion

1. Introduction

Let $\xi > 0$. Let $\widehat{W} = (\widehat{W}^{\widehat{x}}, \widehat{W}^{\widehat{y}}, \widehat{W}^{\widehat{z}})$ be a Brownian motion, and define

$$\begin{aligned}\widehat{x}_t &= \int_0^t e^{(\xi^2/2)(t-s)}\, d\widehat{W}_s^{\widehat{x}} \\ \widehat{y}_t &= \int_0^t e^{(\xi^2/2)(t-s)}\, d\widehat{W}_s^{\widehat{y}} \\ \widehat{z}_t &= \widehat{W}_t^{\widehat{z}}\end{aligned}$$

and

$$\begin{aligned}x &= \widehat{x}\cos\xi\widehat{z} - \widehat{y}\sin\xi\widehat{z} \\ y &= \widehat{x}\sin\xi\widehat{z} + \widehat{y}\cos\xi\widehat{z}.\end{aligned}$$

In Sections 2 and 3, we will show the SDE satisfied by (x, y) and the motivation for the map $(\widehat{x}, \widehat{y}, \widehat{z}) \mapsto (x, y)$. In Sections 4 through 6, we will show that (x, y) satisfies the description of the abstract.

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2. The SDE

The process $(\widehat{x}, \widehat{y}, \widehat{z})$ satisfies the SDE

$$\begin{aligned} d\widehat{x} &= (\xi^2/2)\widehat{x}\,dt + d\widehat{W}^{\widehat{x}} \\ d\widehat{y} &= (\xi^2/2)\widehat{y}\,dt + d\widehat{W}^{\widehat{y}} \\ d\widehat{z} &= d\widehat{W}^{\widehat{z}}. \end{aligned}$$

Define

$$\begin{aligned} W_t^x &= \int_0^t \frac{\cos\xi\widehat{z}_s\,d\widehat{W}_s^{\widehat{x}} - \sin\xi\widehat{z}_s\,d\widehat{W}_s^{\widehat{y}} - \xi y_s\,d\widehat{W}_s^{\widehat{z}}}{\sqrt{1+\xi^2 y_s^2}} \\ W_t^y &= \int_0^t \frac{\sin\xi\widehat{z}_s\,d\widehat{W}_s^{\widehat{x}} + \cos\xi\widehat{z}_s\,d\widehat{W}_s^{\widehat{y}} + \xi x_s\,d\widehat{W}_s^{\widehat{z}}}{\sqrt{1+\xi^2 x_s^2}}. \end{aligned}$$

Then W^x and W^y are Brownian motions with infinitesimal correlation

$$\frac{-\xi^2 xy}{\sqrt{1+\xi^2x^2}\sqrt{1+\xi^2y^2}}$$

and (x, y) satisfies the SDE

$$\begin{aligned} dx &= \sqrt{1+\xi^2y^2}\,dW^x \\ dy &= \sqrt{1+\xi^2x^2}\,dW^y. \end{aligned}$$

3. Isometric action and Riemannian submersion

If we define

$$R_\theta(\widehat{x}, \widehat{y}) = (\widehat{x}\cos\xi\theta - \widehat{y}\sin\xi\theta,\ \widehat{x}\sin\xi\theta + \widehat{y}\cos\xi\theta)$$

and

$$A_\theta(\widehat{x}, \widehat{y}, \widehat{z}) = (R_\theta(\widehat{x}, \widehat{y}), \widehat{z} - \theta),$$

then A is an isometric action on $\mathbb{R}^3$ whose orbits are helices. If we then parametrize the two-dimensional quotient of $\mathbb{R}^3$ by this action by the intercept of the orbit of $(\widehat{x}, \widehat{y}, \widehat{z})$ with the $(\widehat{x}, \widehat{y})$ plane at the point $A_{\widehat{z}}(\widehat{x}, \widehat{y}, \widehat{z})$, then the projection map is given by

$$\varphi(\widehat{x}, \widehat{y}, \widehat{z}) = R_{\widehat{z}}(\widehat{x}, \widehat{y}) = (\widehat{x}\cos\xi\widehat{z} - \widehat{y}\sin\xi\widehat{z},\ \widehat{x}\sin\xi\widehat{z} + \widehat{y}\cos\xi\widehat{z}) = (x, y).$$

The Euclidean dual metric of $\mathbb{R}^3$ is pushed forward by φ, via φ^* pulling back covectors, to the dual metric on $\mathbb{R}^2$ given by the matrix

$$[\varphi^*]\cdot[\varphi^*]^T$$

which is

$$\begin{bmatrix} 1+\xi^2y^2 & -\xi^2xy \\ -\xi^2xy & 1+\xi^2x^2 \end{bmatrix}.$$

That is, $\varphi^* : T^*\mathbb{R}^2 \to T^*\mathbb{R}^3$ respects this dual metric on $\mathbb{R}^2$ and the Euclidean dual metric on $\mathbb{R}^3$. It follows that φ_* is an isometry from the orthogonal complement of its kernel in $T\mathbb{R}^3$ onto $T\mathbb{R}^2$. So φ is a Riemannian submersion. See Chapter 9 of Besse (1987). A direct calculation shows that the Gaussian curvature of $\mathbb{R}^2$ with this dual metric is positive. The unusual property of φ as a nontrivial dimension-reducing submersion, that it respects the dual metrics of its codomain and domain, is due to its construction as the projection map for the quotient of a Riemannian manifold by an isometric action.

4. The Markov property

Following Section 6 of Chapter 10 of Dynkin (1965), we will show that (x, y) is a Markov process and derive its transition density. We will write $\widehat{\alpha}$ or $\widehat{\beta}$ to represent values of $(\widehat{x}, \widehat{y}, \widehat{z})$, and we will write α or β to represent values of (x, y), where convenient.

The process $(\widehat{x}, \widehat{y}, \widehat{z})$ is Markovian and time-homogeneous. See Section 8 of Chapter 4 of Ikeda and Watanabe (1989) and Chapter 7 of Øksendal (2003). Let $\widehat{p}(t, \widehat{\alpha}, \widehat{\beta})$ be its transition density. We define the Borel measures $\nu_t^{\widehat{\alpha}}$ on $\mathbb{R}^2$ by

$$\nu_t^{\widehat{\alpha}}(\Gamma) = \int_{\varphi^{-1}(\Gamma)} \widehat{p}(t, \widehat{\alpha}, \widehat{\beta})\, d\widehat{\beta}.$$

In order to show that $\nu_t^{\widehat{\alpha}}$ is absolutely continuous with respect to the Lebesgue measure, we define $\psi(\widehat{\beta}) = A_{-\widehat{\beta}_3}(\widehat{\beta}_1, \widehat{\beta}_2, 0)$, which is a diffeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ with $\det \psi^* = 1$ satisfying

$$(\varphi \circ \psi)(\widehat{\beta}_1, \widehat{\beta}_2, \widehat{\beta}_3) = (\widehat{\beta}_1, \widehat{\beta}_2).$$

So if the Lebesgue measure $|\Gamma|$ of Γ is zero, then

$$|\varphi^{-1}(\Gamma)| = |\psi^{-1}(\varphi^{-1}(\Gamma))| = |(\varphi \circ \psi)^{-1}(\Gamma)| = |\Gamma \times \mathbb{R}_{\widehat{\beta}_3}| = 0$$

so that $\nu_t^{\widehat{\alpha}}(\Gamma) = 0$. Therefore $\nu_t^{\widehat{\alpha}}$ has the density

$$\widetilde{p}(t, \widehat{\alpha}, \beta) = \lim_{\Gamma \searrow \beta} \frac{1}{|\Gamma|} \int_{\varphi^{-1}(\Gamma)} \widehat{p}(t, \widehat{\alpha}, \widehat{\beta})\, d\widehat{\beta}$$

where the sets Γ are balls centered at β.

In the remainder of this section, we show that $\widetilde{p}(t, \widehat{\alpha}, \beta) = \widetilde{p}(t, \widehat{\alpha}', \beta)$ if $\varphi(\widehat{\alpha}) = \varphi(\widehat{\alpha}')$, so that $(x, y) = \varphi(\widehat{x}, \widehat{y}, \widehat{z})$ is Markovian and time-homogeneous with transition density

$$p(t, \alpha, \beta) = \widetilde{p}(t, \widehat{\alpha}, \beta)$$

for any $\widehat{\alpha} \in \varphi^{-1}(\alpha)$, as in Theorem 10.13 of Dynkin (1965).

Let $v_t = \mathrm{var}(\widehat{x}_t) = \mathrm{var}(\widehat{y}_t) = (1/\xi^2)(\exp(\xi^2 t) - 1)$. Define $c_t = \exp(\xi^2 t/2)$ and $f_t(\widehat{\alpha}) = (c_t\widehat{\alpha}_1, c_t\widehat{\alpha}_2, \widehat{\alpha}_3)$, and define the inner product $\mu(t)$ on $\mathbb{R}^3$ by the matrix

$$\begin{bmatrix} 1/v_t & 0 & 0 \\ 0 & 1/v_t & 0 \\ 0 & 0 & 1/t \end{bmatrix}.$$

Then

$$\widehat{p}(t,\widehat{\alpha},\widehat{\beta}) = \frac{1}{\sqrt{(2\pi)^3 v_t v_t t}} \exp\left(-\tfrac{1}{2}|\widehat{\beta} - f_t(\widehat{\alpha})|^2_{\mu(t)}\right).$$

So

$$\begin{aligned}
\widetilde{p}(t,\widehat{\alpha},\beta) &= \lim_{\Gamma\searrow\beta} \frac{1}{|\Gamma|} \int_{\varphi^{-1}(\Gamma)} \widehat{p}(t,\widehat{\alpha},\widehat{\beta})\, d\widehat{\beta} \\
&= \lim_{\Gamma\searrow\beta} \frac{1}{|\Gamma|} \int_{\psi^{-1}(\varphi^{-1}(\Gamma))} \widehat{p}(t,\widehat{\alpha},\psi(\widehat{\beta}))\, d\widehat{\beta} \\
&= \lim_{\Gamma\searrow\beta} \frac{1}{|\Gamma|} \int_{\Gamma\times\mathbb{R}_{\widehat{\beta}_3}} \widehat{p}(t,\widehat{\alpha},\psi(\widehat{\beta}_1,\widehat{\beta}_2,\widehat{\beta}_3))\, d\widehat{\beta}_1\, d\widehat{\beta}_2\, d\widehat{\beta}_3 \\
&= \int_{\mathbb{R}} \widehat{p}(t,\widehat{\alpha},\psi(\beta_1,\beta_2,\theta))\, d\theta \\
&= \int_{\mathbb{R}} \widehat{p}(t,\widehat{\alpha},A_{-\theta}(\beta_1,\beta_2,0))\, d\theta \\
&= \frac{1}{\sqrt{(2\pi)^3 v_t v_t t}} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}|A_{-\theta}(\beta_1,\beta_2,0) - f_t(\widehat{\alpha})|^2_{\mu(t)}\right) d\theta \\
&= \frac{1}{\sqrt{(2\pi)^3 v_t v_t t}} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}|(\beta_1,\beta_2,0) - A_{\theta} f_t(\widehat{\alpha})|^2_{\mu(t)}\right) d\theta
\end{aligned}$$

since A_θ is an isometry of $\mathbb{R}^3_{\mu(t)}$ for every θ.

Now choose $\widehat{\alpha}' \in \varphi^{-1}(\varphi(\widehat{\alpha}))$. Then there exists $\lambda \in \mathbb{R}$ such that $\widehat{\alpha}' = A_\lambda(\widehat{\alpha})$. Note that A_λ and f_t commute. So the norm in the integrand of $\widetilde{p}(t,\widehat{\alpha}',\beta)$ is

$$|(\beta_1,\beta_2,0) - A_\theta f_t(A_\lambda(\widehat{\alpha}))|_{\mu(t)} = |(\beta_1,\beta_2,0) - A_{\theta+\lambda} f_t(\widehat{\alpha})|_{\mu(t)}$$

and

$$\begin{aligned}
\widetilde{p}(t,\widehat{\alpha}',\beta) &= \frac{1}{\sqrt{(2\pi)^3 v_t v_t t}} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}|(\beta_1,\beta_2,0) - A_{\theta+\lambda} f_t(\widehat{\alpha})|^2_{\mu(t)}\right) d\theta \\
&= \frac{1}{\sqrt{(2\pi)^3 v_t v_t t}} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}|(\beta_1,\beta_2,0) - A_{\theta} f_t(\widehat{\alpha})|^2_{\mu(t)}\right) d\theta \\
&= \widetilde{p}(t,\widehat{\alpha},\beta).
\end{aligned}$$

So (x,y) is Markovian and time-homogeneous with transition density

$$\begin{aligned}
p(t,\alpha,\beta) &= \widetilde{p}(t,(\alpha_1,\alpha_2,0),\beta) \\
&= \frac{1}{\sqrt{(2\pi)^3 v_t v_t t}} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}(|\beta - c_t R_\theta \alpha|^2/v_t + \theta^2/t)\right) d\theta.
\end{aligned}$$

5. Gaussian distribution

The density of (x_t, y_t) is

$$p(t,0,\beta) = \frac{1}{\sqrt{(2\pi)^3 v_t v_t t}} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}(|\beta|^2/v_t + \theta^2/t)\right)\, d\theta$$
$$= \frac{1}{\sqrt{(2\pi)^2 v_t v_t}} \exp\left(-\tfrac{1}{2}|\beta|^2/v_t\right).$$

So (x_t, y_t) is Gaussian with covariance matrix

$$\begin{bmatrix} v_t & 0 \\ 0 & v_t \end{bmatrix}.$$

6. The martingale property

A direct calculation shows that

$$\int_{\mathbb{R}^2} \beta_k\, p(t,\alpha,\beta)\, d\beta = \alpha_k$$

using the fact that $E(\cos w\gamma) = \exp(-w^2/2)$ if γ is a standard normal variable. We similarly observe that $\int |\beta_k|\, p(t,\alpha,\beta)\, d\beta$ is finite for every $t > 0$ and every $\alpha \in \mathbb{R}^2$. So if we define the bounded functions $f_n^k(\beta) = \min(\max(\beta_k, -n), n)$ so that f_n^k converges to the map $\beta \mapsto \beta_k$ as n increases without bound, then a direct application of the dominated convergence theorem and the conditional dominated convergence theorem shows that

$$E((x_{t+h}, y_{t+h})|\mathcal{F}_t) = \lim E^{(x_t,y_t)}(f_n^1(x_h,y_h), f_n^2(x_h,y_h)) = (x_t, y_t)$$

where $\mathcal{F}$ is the filtration generated by (x, y). Since $E(|(x_t, y_t)|)$ is obviously finite, (x, y) is therefore a martingale.

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